NORTH-HOLLAND

Criterion of High-Codimensional Bifurcations With Several Pairs of Purely Imaginary Eigenvalues

Zhi-Ming Wu and Wei-min Liu*

Department of Mathematical Sciences
Indiana University Purdue University Indianapolis
402 North Blackford Street
Indianapolis, Indiana 46202-3216

Submitted by Chi-Kwong Li

ABSTRACT

This paper presents a criterion for high-codimensional bifurcations with several pairs of purely imaginary eigenvalues in terms of the properties of coefficients of characteristic polynomials instead of those of eigenvalues. It uses the Routh-Hurwitz determinants and is convenient in the symbolic analysis of complicated systems. © 1997 Elsevier Science Inc.

1. INTRODUCTION

When parameters in a system of differential equations are varied, the qualitative structure of its solutions may change. These changes are called bifurcations [1]. For example, the Hopf bifurcation is a route for a nonlinear system to change from steady states to persistent oscillations. The traditional theory about Hopf bifurcation is stated in terms of the eigenvalues of the Jacobian matrix. Real-world problems usually include many parameters (e.g., [3], an application in epidemic models). To analyze such systems, one usually resorts to symbolic computations [4]. For complicated systems, however, symbolic computations seldom yield meaningful results about the eigenvalues. For systematic analysis with symbolic calculations, it is ideal to have a

* E-mail: zwu@math.iupui.edu, liu@math.iupui.edu.

655 Avenue of the Americas, New York, NY 10010

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criterion stated in terms of the coefficients of the characteristic polynomial. Liu [5] presents a criterion for the simple Hopf bifurcation where a pair of imaginary eigenvalues passes through the imaginary axis and the other eigenvalues remain in the left half plane. His criterion makes use of the Routh-Hurwitz determinants and does not directly use eigenvalues. It is convenient in many applications. But his criterion does not cover the high-codimensional bifurcations [1, 2]. In this paper, we consider an m-codimensional bifurcation: all eigenvalues of the Jacobian matrix have negative real parts before the bifurcation, and m pairs of imaginary eigenvalues are on the imaginary axis while all other eigenvalues remain in the left half plane when the bifurcation occurs.

2. NOTATION

We use the notation in [1, 5] with minor changes. Consider a system

$$\dot{x} = f_\mu(x), \qquad x \in R^n, \quad \mu \in R^m, \tag{1}$$

with an equilibrium x_0 for $\mu \in \Omega$, where Ω is a simply connected open region in R^m and f is sufficiently smooth in both x and μ. We are interested in the situation that the Jacobian matrix $J(\mu) = D_x f_\mu(x_0)$ is stable, i.e., all its eigenvalues have negative real parts for $\mu \in \Omega$, and that $J(\mu_0)$ has m pairs of purely imaginary eigenvalues and $n - 2m$ eigenvalues with negative real parts for a boundary point μ_0 of Ω.

Let us denote the characteristic polynomial of the Jacobian matrix $J(\mu)$ by

$$p_n(\lambda; \mu) = \det[\lambda I_n - J(\mu)] = p_0(\mu) + p_1(\mu)\lambda + \cdots + p_n(\mu)\lambda^n, \tag{2}$$

where every $p_i(\mu)$ is a smooth function of μ, $p_n(\mu) = 1$, and we can restrict ourselves to the case $p_0(\mu) > 0$ because there is no nonnegative real zero of $p_n(\lambda; \mu)$. The kth Routh-Hurwitz matrix of $p_n(\lambda; \mu)$ or $J(\mu)$ is defined by

$$L_k(\mu) = \begin{pmatrix} p_1(\mu) & p_0(\mu) & \cdots & 0 \\ p_3(\mu) & p_2(\mu) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ p_{2k-1}(\mu) & p_{2k-2}(\mu) & \cdots & p_k(\mu) \end{pmatrix},$$

where $p_i(\mu) = 0$ if $i < 0$ or $i > n$. According to [6], the Routh-Hurwitz criterion can be stated as that all zeros of the polynomial $p_n(\lambda; \mu)$ have negative real parts if and only if $p_0(\mu) > 0$ and the n Routh-Hurwitz determinants are positive:

$$D_1(\mu) = \det L_1(\mu) = p_1(\mu) > 0,$$

$$D_2(\mu) = \det L_2(\mu) = \det\begin{pmatrix} p_1(\mu) & p_0(\mu) \\ p_3(\mu) & p_2(\mu) \end{pmatrix} > 0,$$

$$\vdots$$

$$D_n(\mu) = \det L_n(\mu) > 0.$$

Since $D_n(\mu) = p_n(\mu)D_{n-1}(\mu)$ and for characteristic polynomials $p_n(\mu) = 1$, the Routh-Hurwitz conditions can be expressed as

$$p_0 > 0, \qquad D_1 > 0, \qquad D_2 > 0, \ \ldots, \qquad D_{n-1} > 0. \tag{3}$$

3. CRITERION

Before presenting our criterion, we need a lemma.

LEMMA 1. *Assume the Routh-Hurwitz determinants of the polynomial*

$$q_{n-2}(\lambda) = q_0 + q_1\lambda + \cdots + q_{n-2}\lambda^{n-2}$$

are $Q_1, Q_2, \ldots, Q_{n-2}$. *If* $p_n(\lambda) = q_{n-2}(\lambda)(\lambda^2 + \beta)$, *then the Routh-Hurwitz determinants of* $p_n(\lambda)$ *are*

$$D_1 = \beta Q_1, \quad D_2 = \beta^2 Q_2, \ldots, \quad D_{n-2} = \beta^{n-2} Q_{n-2}, \quad D_{n-1} = D_n = 0.$$

Proof. Let $p_n(\lambda) = p_0 + p_1\lambda + \cdots + p_n\lambda^n$. We have $p_i = \beta q_i + q_{i-2}$ $(i = 0, 1, \ldots, n)$, where $q_i = 0$ if $i < 0$ or $i > n - 2$. The nth Routh-Hurwitz matrix of $p_n(\lambda)$ can be written as

$$L_n = \begin{pmatrix} \beta q_1 & \beta q_0 & 0 & 0 & \cdots & 0 \\ \beta q_3 + q_1 & \beta q_2 + q_0 & \beta q_1 & \beta q_0 & \cdots & 0 \\ \beta q_5 + q_3 & \beta q_4 + q_2 & \beta q_3 + q_1 & \beta q_2 + q_0 & \cdots & 0 \\ \beta q_7 + q_5 & \beta q_6 + q_4 & \beta q_5 + q_3 & \beta q_4 + q_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & q_{n-2} \end{pmatrix}.$$

The Routh-Hurwitz determinants can be calculated as

$$D_1 = \beta q_1 = \beta Q_1,$$

$$D_2 = \begin{vmatrix} \beta q_1 & \beta q_0 \\ \beta q_3 + q_1 & \beta q_2 + q_0 \end{vmatrix} = \beta \begin{vmatrix} q_1 & q_0 \\ \beta q_3 + q_1 & \beta q_2 + q_0 \end{vmatrix}.$$

$$= \beta \begin{vmatrix} q_1 & q_0 \\ \beta q_3 & \beta q_2 \end{vmatrix} = \beta^2 Q_2.$$

Using the properties of the determinant, we can obtain other equalities. D_{n-1} is zero because we repeatedly use a row to eliminate the second terms in the next row and the last row becomes a zero row. Since $D_n = D_{n-1}$, D_n is also zero. ■

THEOREM 1. *Let the characteristic polynomial of an $n \times n$ matrix J be*

$$p_n(\lambda) = \det(\lambda I - J) = p_0 + p_1\lambda + \cdots + p_n\lambda^n \qquad (p_n = 1).$$

If J has m ($1 < 2m \leqslant n$) pairs of purely imaginary eigenvalues and $n - 2m$ eigenvalues with negative parts, then

(a) $p_0 > 0$, $p_i \geqslant 0$ ($i = 1, \ldots, n-1$); when $n > 2m$, $p_i > 0$ for $i = 0, 1, \ldots, n$;

(b) the $2m$ higher-order Routh-Hurwitz determinants of $p_n(\lambda)$ are equal to zero, i.e., $D_n = D_{n-1} = \cdots = D_{n-2m+1} = 0$;

(c) when $n > 2m$, the remaining $n - 2m$ lower-order Routh-Hurwitz determinants are positive, i.e., $D_{n-2m} > 0, \ldots, D_1 > 0$.

REMARK. Since $D_n = p_n D_{n-1} = D_{n-1}$, in applications it is unnecessary to verify $D_n = 0$ when $D_{n-1} = 0$. Moreover, when $n > 2m$, $D_1 = p_1$. Thus, $p_1 > 0$ implies that $D_1 > 0$.

Proof. If J has m pairs of purely imaginary eigenvalues and $(n - 2m)$ eigenvalues with negative real parts, then

$$p_n(\lambda) = q_{n-2m}(\lambda) \prod_{i=1}^{m} (\lambda^2 + \beta_i) \qquad (\beta_i > 0, \quad i = 1, \dots, m).$$

If $n = 2m$, then $q_0(\lambda) = 1$, all odd-degree coefficients of $p_n(\lambda)$ are zero, and all its even-degree coefficients are positive. If $n > 2m$, then $q_{n-2m}(\lambda)$ is a stable polynomial of degree at least one, all its coefficients and Routh-Hurwitz determinants are positive. Thus, all coefficients of $p_n(\lambda)$ are positive.

Repeatedly using Lemma 1, we can show that

$$D_1 = \beta_1 \cdots \beta_m Q_1 > 0,$$

$$D_2 = \beta_1^2 \cdots \beta_m^2 Q_2 > 0,$$

$$\vdots$$

$$D_{n-2m} = \beta_1^{n-2m} \cdots \beta_m^{n-2m} Q_{n-2m} > 0,$$

$$D_{n-2m+1} = 0, \ \dots, \qquad D_{n-1} = 0, \qquad D_n = 0. \qquad \blacksquare$$

Note that the converse of Theorem 1 is not true. Consider the following example.

EXAMPLE. Let $p_5(\lambda) = (\lambda + \gamma)(\lambda^2 - 2\alpha\lambda + \beta)(\lambda^2 + 2\alpha\lambda + \beta)$, where $\gamma > 0$, $\beta > 0$, and $0 < \alpha < \sqrt{\beta/2}$. Therefore, $p_5(\lambda)$ does not have any purely imaginary zeros. All coefficients of $p_5(\lambda)$ are positive, and

$$L_5 = \begin{pmatrix} \beta^2 & \gamma\beta^2 & 0 & 0 & 0 \\ 2\beta-4\alpha^2 & \gamma(2\beta-4\alpha^2) & \beta^2 & \gamma\beta^2 & 0 \\ 1 & \gamma & 2\beta-4\alpha^2 & \gamma(2\beta-4\alpha^2) & \beta^2 \\ 0 & 0 & 1 & \gamma & 2\beta-4\alpha^2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$p_0 = \gamma\beta^2 > 0, \qquad D_1 = \beta^2 > 0, \qquad D_2 = D_3 = D_4 = D_5 = 0.$$

According to Orlando's formula [6]

$$D_{n-1} = (-1)^{(n-1)n/2} \prod_{i<k}^{1,\dots,n} (\lambda_i + \lambda_k),$$

where λ_i $(i = 1, \dots, n)$ are the zeros of $p_n(\lambda)$. Therefore $D_{n-1} = 0$ if and only if the sum of two zeros of $p_n(\lambda)$ is zero. We consider the case where $p_n(\lambda)$ does not have nonnegative real zeros. The only two possibilities for $D_{n-1} = 0$ are (i) that there is a pair of purely imaginary zeros and (ii) that there are two pairs of imaginary zeros $\alpha \pm \omega i$ and $-\alpha \pm \omega i$ $(\alpha > 0, \omega > 0)$. When $\mu \in \Omega$, the matrix J is stable, i.e., all its eigenvalues have negative real parts. When the eigenvalues change continuously with the parameters, at $\mu_0 \in \partial\Omega$, situation (ii) cannot occur, that is, the eigenvalues cannot jump to the right half plane without crossing the imaginary axis. Therefore, we have the following result.

THEOREM 2. *Assume $J(\mu)$ is the Jacobian matrix of the system* (1). *If in a simply connected open region Ω of μ in R^m and at the equilibrium x_0, the characteristic polynomial $p_n(\lambda; \mu)$ of $J(\mu)$ is stable, i.e., $p_0(\mu) > 0$, $D_1(\mu) > 0, \dots, D_{n-1}(\mu) > 0$, and if μ_0 is a boundary point of Ω where $p_0(\mu_0) > 0$, $p_i(\mu_0) \geqslant 0$, $D_{n-1}(\mu_0) = \cdots = D_{n-2m+1}(\mu_0) = 0$, and for $n > 2m$ one has $D_{n-2m}(\mu_0) > 0, \dots, D_1(\mu_0) > 0$, then there is an m-codimensional bifurcation with m pairs of purely imaginary eigenvalues at $\mu = \mu_0$.*

The next question in bifurcation theory is what will happen when μ passes through μ_0. We mainly consider the important special case of bifurcation with two pairs of purely imaginary eigenvalues. We can use following two lemmas to prove our results summarized in Theorem 3.

LEMMA 2. *Suppose the polynomial* $p_n(\lambda;\mu)$ *can be factorized as*

$$p_n(\lambda;\mu) = q_{n-2m}(\lambda;\mu) r_{2m}(\lambda;\mu), \tag{4}$$

where the leading coefficients of p_n, q_{n-2m}, *and* r_{2m} *are all equal to* 1, *and* $p_n(\lambda;\mu)$ *is a stable polynomial for* $\mu \in \Omega$ *in* R^m. *Further assume that in a neighborhood of* $\mu_0 \in \partial\Omega$, $q_{n-2m}(\lambda;\mu)$ *is a stable polynomial, and* $r_{2m}(\lambda;\mu) = \prod_{i=1}^{m}(\lambda^2 + 2\alpha_i\lambda + \beta_i)$ *where* α_i *and* β_i *are smooth functions of* μ. *All* β_i $(i = 1, \ldots, m)$ *are positive in the neighborhood of* μ_0, *and all* $\alpha_i = 0$ *when* $\mu = \mu_0$. *Then in a neighborhood of* μ_0, *the Routh-Hurwitz determinant of* $p_n(\lambda;\mu)$ *is*

$$D_{n-1}(\mu) = C(\mu) Q_{n-2m}(\mu) R_{2m}(\mu), \tag{5}$$

where $C(\mu) > 0$, *and* $Q_{n-2m}(\mu)$ *and* $R_{2m}(\mu)$ *are the Routh-Hurwitz determinants of* $q_{n-2m}(\lambda;\mu)$ *and* $r_{2m}(\lambda;\mu)$.

Proof. Let $\lambda_1, \lambda_2, \ldots, \lambda_{n-2m}$ be the zeros of $q_{n-2m}(\lambda;\mu)$, and $\lambda_{n-2m+1}, \ldots, \lambda_n$ be the zeros of $r_{2m}(\lambda;\mu)$. By Orlando's formula,

$$\begin{aligned} D_{n-1}(\mu) &= (-1)^{n(n-1)/2} \prod_{i<k}^{1,\ldots,n} (\lambda_i + \lambda_k) \\ &= \prod_{k=n-2m+1,\ldots,n}^{1,\ldots,n-2m} (\lambda_i + \lambda_k) \\ &\quad \times \left[(-1)^{(n-2m)(n-2m-1)/2} \prod_{i<k}^{i=1,\ldots,n-2m} (\lambda_i + \lambda_k) \right] \\ &\quad \times \left[(-1)^{2m(2m-1)/2} \prod_{i<k}^{n-2m+1,\ldots,n} (\lambda_i + \lambda_k) \right]. \end{aligned} \tag{6}$$

The product in the first pair of brackets is equal to $Q_{n-2m}(\mu)$. The product in the second pair of brackets is nothing but $R_{2m}(\mu)$. What we need to show now is that the product before the brackets is positive. When λ_i is a negative real number and λ_k is a purely imaginary number,

$$(\lambda_i + \lambda_k)(\lambda_i + \bar{\lambda}_k) = |\lambda_i|^2 + |\lambda_k|^2 > 0.$$

When λ_i is an imaginary number with negative real part and λ_k is a purely imaginary number,

$$(\lambda_i + \lambda_k)(\bar{\lambda}_i + \bar{\lambda}_k) = |\lambda_i + \lambda_k|^2 > 0.$$

So the lemma is proved. ∎

COROLLARY. *Under the assumptions of Lemma 2, if $m = 2$, then*

$$p_n(\lambda; \mu) = G(\mu) R_4(\mu), \tag{7}$$

where $G(\mu) > 0$ in a neighborhood of $\mu_0 \in \partial\Omega$ and

$$R_4(\mu) = 4\alpha_1\alpha_2\left[(\beta_1 - \beta_2)^2 + 4(\alpha_1 + \alpha_2)(\beta_2\alpha_1 + \beta_1\alpha_2)\right]. \tag{8}$$

Proof. Since $q_{n-2m}(\lambda; \mu)$ is stable, $G(\mu) = C(\mu)Q_{n-4}(\mu) > 0$. The only thing one needs to verify is that $R_4(\mu)$ can be written in the form of (8). ∎

LEMMA 3. *Under the assumption of Lemma 2, if $m = 2$, then in a neighborhood of μ_0,*

$$D_{n-2}(\mu) = A(\mu)\beta_2^{n-2}\alpha_1 + B(\mu)\beta_1^{n-2}\alpha_2 + o(\alpha_1, \alpha_2) \tag{9}$$

and

$$D_{n-3}(\mu) = A(\mu)\beta_2^{n-3}\alpha_1 + B(\mu)\beta_1^{n-3}\alpha_2 + o(\alpha_1, \alpha_2), \tag{10}$$

where $A(\mu)$ and $B(\mu)$ are positive and $o(\alpha_1, \alpha_2)$ represents high-order terms of α_1 and α_2.

Proof. $D_{n-2}(\mu)$ and $D_{n-3}(\mu)$ are polynomials in $\alpha_1 = \alpha_1(\mu)$ and $\alpha_2 = \alpha_2(\mu)$. For convenience of the proof, we denote them by $D_{n-2}(\alpha_1, \alpha_2)$ and $D_{n-3}(\alpha_1, \alpha_2)$. Their Maclaurin expansion can be grouped as

$$D_{n-2}(\alpha_1, \alpha_2) = \alpha_1 f_1(\alpha_1) + \alpha_2 f_2(\alpha_2) + \alpha_1\alpha_2 f_3(\alpha_1, \alpha_2) \tag{11}$$

and

$$D_{n-3}(\alpha_1, \alpha_2) = \alpha_1 g_1(\alpha_1) + \alpha_2 g_2(\alpha_2) + \alpha_1 \alpha_2 g_3(\alpha_1, \alpha_2) \quad (12)$$

where f_1, f_2, f_3, g_1, g_2, and g_3 are polynomials. When $\alpha_2 = 0$, (11) and (12) become

$$D_{n-2}(\alpha_1, 0) = \alpha_1 f_1(\alpha_1) \quad \text{and} \quad D_{n-3}(\alpha_1, 0) = \alpha_1 g_1(\alpha_1).$$

On the other hand, by Lemma 1,

$$D_{n-2}(\alpha_1, 0) = \beta_2^{n-2} H_{n-2}(\alpha_1), \qquad D_{n-3}(\alpha_1, 0) = \beta_2^{n-3} H_{n-3}(\alpha_1),$$

where H_{n-2} and H_{n-3} are the Routh-Hurwitz determinants of the polynomial

$$h_{n-2}(\lambda; \alpha_1) = (\lambda^2 + 2\alpha_1 \lambda + \beta_1) q_{n-4}(\lambda).$$

Thus, $H_{n-2}(\alpha_1) = H_{n-3}(\alpha_1)$. The coefficients of $h_{n-2}(\lambda; \alpha_1)$ are

$$h_i = \beta_1 q_i + 2\alpha_1 q_{i-1} + q_{i-2}.$$

Therefore, the $(n-3)$th Routh-Hurwitz matrix of $h_{n-2}(\lambda; \alpha_1)$ can be written as

$$L_{n-3}(\alpha_1) = M_{n-3} + 2\alpha_1 N_{n-3},$$

where M_{n-3} is a matrix of rank $n-4$, and N_{n-3} is s nonsingular matrix (see [5]). Note that

$$H_{n-3}(0) = \det L_{n-3}(0) = \det M_{n-3} = 0;$$

hence the Maclaurin expansion of $H_{n-3}(\alpha_1)$ is

$$H_{n-3}(\alpha_1) = C\alpha_1 + o(\alpha_1).$$

According to Lemma 2 of [5], $C \neq 0$. Thus,

$$D_{n-2}(\alpha_1, 0) = \alpha_1 \beta_2^{n-2} C + o(\alpha_1),$$

and

$$D_{n-3}(\alpha_1, 0) = \alpha_1 \beta_2^{n-3} C + o(\alpha_1).$$

When $\alpha_1 > 0$, D_{n-2} and D_{n-3} must be positive. Therefore $C > 0$. Consider the symmetric case where $\alpha_1 = 0$, we can get the similar expansions of $D_{n-2}(0, \alpha_2)$ and $D_{n-3}(0, \alpha_2)$. Combining these results, we have (9) and (10), where $A(\mu)$ and $B(\mu)$ are positive in a neighborhood of μ_0. ■

Now we present our result for the bifurcation with *two pairs of purely imaginary* eigenvalues.

THEOREM 3. *Assume that Ω is a simply connected open region in R^2; for $\mu \in \Omega$ the characteristic polynomial $p_n(\lambda; \mu)$ $(n \geqslant 4)$ of the system* (1) *at the equilibrium x_0 is stable; μ_0 is a boundary point of Ω. There is a bifurcation of two pairs of purely imaginary eigenvalues $-\alpha_j(\mu) \pm \omega_j(\mu)i$, $\{\alpha_j(\mu_0) = 0, \omega_j(\mu_0) > 0, j = 1, 2\}$ if and only if $p_0(\mu_0) > 0$, $D_{n-1}(\mu_0) = D_{n-2}(\mu_0) = D_{n-3}(\mu_0) = 0$, and when $n > 4$, $D_{n-4}(\mu_0) > 0, \dots, D_1(\mu_0) > 0$. When μ leaves $\overline{\Omega}$ passing through μ_0, there are the following different situations*:

(i) $D_{n-1}(\mu) < 0$ *if and only if a pair of the imaginary eigenvalues moves to the right half plane and the other pair returns to the left half plane* $(\alpha_j(\mu) > 0$, $\alpha_{3-j}(\mu) < 0$, $j = 1$ *or* 2) *except for the mirror symmetric case* $\alpha_1 = -\alpha_2 \neq 0$ *and* $\beta_1 = \beta_2$.

(ii) $D_{n-1}(\mu) > 0$, $D_{n-2}(\mu) < 0$, *and* $D_{n-3}(\mu) < 0$ *if and only if the two pairs of the imaginary eigenvalues both move to the right half plane* $(\alpha_j(\mu) < 0$, $j = 1$ *and* 2);

(iii) $D_{n-1}(\mu) > 0$, $D_{n-2}(\mu) > 0$, *and* $D_{n-3}(\mu) > 0$ *if and only if the two pairs of the imaginary eigenvalues both return to the left half plane* $(\alpha_j(\mu) > 0$, $j = 1$ *and* 2);

(iv) $D_{n-1}(\mu) = 0$, $D_{n-2}(\mu) < 0$, *and* $D_{n-3}(\mu) < 0$ *if and only if a pair of the imaginary eigenvalues moves to the right half plane and the other pair remains on the imaginary axis* $(\alpha_j(\mu) < 0$, $\alpha_{3-j}(\mu) = 0$, $j = 1$ *or* 2);

(v) $D_{n-1}(\mu) = 0$, $D_{n-2}(\mu) > 0$, *and* $D_{n-3}(\mu) > 0$ *if and only if a pair of the imaginary eigenvalues returns to the left half plane and the other pair remains on the imaginary axis* $(\alpha_j(\mu) > 0$, $\alpha_{3-j}(\mu) = 0$, $j = 1$ *or* 2);

(vi) $D_{n-1}(\mu) = D_{n-2}(\mu) = D_{n-3}(\mu) = 0$ *if and only if the two pairs of the imaginary eigenvalues remain on the imaginary axis* $(\alpha_j(\mu) = 0$, $j = 1$ *and* 2) *or a pair returns to the left half plane and the other pair moves to the right half plane with mirror symmetry about the imaginary axis* $(\alpha_1(\mu) = -\alpha_2(\mu)$, $\omega_1(\mu) = \omega_2(\mu) > 0)$.

Proof. The condition for a bifurcation with two pairs of purely imaginary eigenvalues $-\alpha_j \pm i\omega_j$ $[\alpha_j(\mu_0) = 0, j = 1, 2]$ is an application of Theorems 1 and 2 to the special case $m = 2$. We note that $\omega_j = \sqrt{\beta_j - \alpha_j^2}$ when there are factors $\lambda^2 + 2\alpha_j\lambda + \beta_j$ $(\beta_j > 0)$.

Using the corollary of Lemma 2, we can show that in a neighborhood of μ_0, the sign of $D_{n-1}(\mu)$ is the sign of $\alpha_1\alpha_2$ when $\beta_1 \neq \beta_2$, and is the sign of $\alpha_1\alpha_2(\alpha_1 + \alpha_2)^2$ when $\beta_1 = \beta_2$.

Thus, $D_{n-1}(\mu) < 0$ if and only if $\alpha_1\alpha_2 < 0$ except for the mirror symmetric case $\alpha_1 = -\alpha_2 \neq 0$ and $\beta_1 = \beta_2$, which is summarized in (vi).

When $D_{n-1}(\mu) > 0$, α_1 and α_2 must have the same sign. According to Lemma 3, $D_{n-2}(\mu)$ and $D_{n-3}(\mu)$ must have the same sign as α_1 and α_2. So (ii) and (iii) can be verified.

When $D_{n-1}(\mu) = 0$, there are the following possibilities: one and only one of α_1 and α_2 is zero (in this case D_{n-2} and D_{n-3} have the same sign as the nonzero α_j); both α_1 and α_2 are zero; $\alpha_1 = -\alpha_2 \neq 0$ and $\beta_1 = \beta_2$. They are summarized in (iv)–(vi). ■

REMARK. When $f_\mu(x)$ depends on μ analytically, the points of μ where $D_i(\mu)$ is zero must be isolated because $D_i(\mu) > 0$ for $\mu \in \Omega$. Situations (iv)–(vi) do not occur. There are only three possibilities: (i)–(iii).

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